[09-01-21-T11]

Approximating square roots - Newton's method

Newton's method

■ Derive formulae for $\sqrt{N}$

Suppose $\sqrt{N} = a + \epsilon$, $0 \le \epsilon << 1$.

$$\sqrt{N} = a + \epsilon$$

$$N = (a + \epsilon)^2$$

$$= a^2 + 2\,a\,\epsilon + \epsilon^2$$

$$N \approx a^2 + 2\,a\,\epsilon$$

$$\Longrightarrow \epsilon \approx \frac{N-a^2}{2\,a}$$

$$\Longrightarrow \sqrt{N} \approx a + \frac{N-a^2}{2\,a}$$

$$\approx \frac{2\,a^2+N-a^2}{2\,a}$$

$$\approx \frac{a^2+N}{2\,a}$$

$$\sqrt{N} \approx \frac{a+\frac{N}{a}}{2}$$

In practice, this formula is used recursivly. The immediate value of a becomes a for the next computation. The computation is continued until the desired degree of accuracy is obtained. The reursive formula is

$$\sqrt{N} \approx a_k \quad \frac{a_{k-1}+\frac{N}{a_{k-1}}}{2}$$

Example 1

Approximate the square root of 7 to 4 decimal places. Note that we will round intermediate results to 5 decimal digits.

Since $2^2 = 4$, $3^2 = 9$, choose $a = 2$.

$$\frac{2+\frac{7}{2}}{2} = \frac{11}{4} \approx 2.75000$$

$$\frac{2.75 + \frac{7}{2.75}}{2} = 2.64773$$

$$\frac{2.64773 + \frac{7}{2.64773}}{2} = 2.64575$$

Since the this result does not differ from the immediately preceding result in the fourth decimal digit, the approximation is good to four decimal digits.